A Note on Positive Energy Theorem for Spaces with Asymptotic SUSY Compactification

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Abstract

We extend the positive mass theorem in [D] to the Lorentzian setting. This includes the original higher dimensional Positive Energy Theorem whose spinor proof is given in [Wi1] and [PT] for dimension 4 and in [Z1] for dimension 5.

1 Introduction and statement of the result

In this note, we formulate and prove the Lorentzian version of the positive mass theorem in [D]. There we prove a positive mass theorem for spaces which asymptotically approach the product of a flat Euclidean space with a compact manifold which admits a nonzero parallel spinor (such as a Calabi-Yau manifold or any special honolomy manifold except the quaternionic Kähler). This is motivated by string theory, especially the recent work [HHM]. The application of the positve mass theorem of [D] to the study of stability of Ricci flat manifolds is discussed in [DWW].

In general relativity, a spacetime is modeled by a Lorentzian 4-manifold $(N,\ g)$ together with an energy-momentum tensor T satisfying Einstein equation

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta}. \tag{1.1}$$

The positive energy theorem [SY1], [Wi1] says that an isolated gravitational system with nonnegative local matter density must have nonnegative total energy, measured at spatial infinity. More precisely, one considers a complete oriented spacelike hypersurface M of N satisfying the following two conditions:

a). M *is asymptotically flat*, that is, there is a compact set K in M such that $M - K$ is the disjoint union of a finite number of subsets $M_1, \ldots, M_k$ and each M_l is diffeomorphic to $(\mathbb{R}^3 - B_R(0))$. Moreover, under this diffeomorphism, the metric of M_l is of the form

$$g_{ij} = \delta_{ij} + O(r^{-\tau}), \quad \partial_k g_{ij} = O(r^{-\tau-1}), \quad \partial_k \partial_l g_{ij} = O(r^{-\tau-2}). \tag{1.2}$$

Furthermore, the second fundamental form h_{ij} of M in N satisfies

$$h_{ij} = O(r^{-\tau-1}), \quad \partial_k h_{ij} = O(r^{-\tau-2}). \tag{1.3}$$

Here $\tau > 0$ is the asymptotic order and r is the Euclidean distance to a base point.

b). *M has nonnegative local mass density*: for each point $p \in M$ and for each timelike vector e_0 at p, $T(e_0,\ e_0) \geq 0$ and $T(e_0,\ \cdot)$ is a nonspacelike co-vector. This implies the dominant energy condition

$$T^{00} \geq |T^{\alpha\beta}|, \quad T^{00} \geq (-T_{0i}T^{0i})^{\frac{1}{2}}. \tag{1.4}$$

The total energy (the ADM mass) and the total (linear) momentum of M can then be defined as follows [ADM], [PT] (for simplicity we suppress the dependence here on l (the end M_l))

$$\begin{aligned} E &= \lim_{R\to\infty} \frac{1}{4\omega_n} \int_{S_R} (\partial_i g_{ij} - \partial_j g_{ii}) * dx_j, \\ P_k &= \lim_{R\to\infty} \frac{1}{4\omega_n} \int_{S_R} 2(h_{jk} - \delta_{jk} h_{ii}) * dx_j \end{aligned} \tag{1.5}$$

Here ω_n denotes the volume of the $n-1$ sphere and S_R the Euclidean sphere with radius R centered at the base point.

Theorem 1.1 (Schoen-Yau, Witten) *With the assumptions as above and assuming that M is spin, one has*

$$E - |P| \geq 0$$

on each end M_l. Moreover, if $E = 0$ for some end M_l, then M has only one end and N is flat along M.

Now, according to string theory [CHSW], our universe is really ten dimensional, modelled on $\mathbb{R}^{3,1} \times X$ where X is a Calabi-Yau 3-fold. This is the so called Calabi-Yau compactification, which motivates the spaces we now consider.

Thus, we consider a Lorentzian manifold N (with signature $(-,+,\cdots,+)$) of $\dim N = n+1$, with a energy-momentum tensor satisfying the Einstein equation. Then let M be a complete oriented spacelike hypersurface in N. Furthermore the Riemannian manifold (M^n, g) with g induced from the Lorentzian metric decomposes $M = M_0 \cup M_\infty$, where M_0 is compact as before but now $M_\infty \simeq (\mathbb{R}^k - B_R(0)) \times X$ for some radius $R > 0$ and X a compact simply connected spin manifold which admits a nonzero parallel spinor. Moreover the metric on M_∞ satisfies

$$g = \mathring{g} + u, \quad \mathring{g} = g_{\mathbb{R}^k} + g_X, \quad u = O(r^{-\tau}), \quad \mathring{\nabla} u = O(r^{-\tau-1}), \quad \mathring{\nabla}\mathring{\nabla} u = O(r^{-\tau-2}), \tag{1.6}$$

and the second fundamental form h of M in N satisfies

$$h = O(r^{-\tau-1}), \quad \mathring{\nabla} h = O(r^{-\tau-2}). \tag{1.7}$$

Here $\mathring{\nabla}$ is the Levi-Civita connection of $\mathring{g}$ (extended to act on all tensor fields), $\tau > 0$ is the asymptotical order.

The total energy and total momentum for such a space can then be defined by

$$\begin{aligned} E &= \lim_{R\to\infty} \frac{1}{4\omega_k vol(X)} \int_{S_R\times X} (\partial_i g_{ij} - \partial_j g_{aa}) * dx_j dvol(X), \\ P_k &= \lim_{R\to\infty} \frac{1}{4\omega_k vol(X)} \int_{S_R\times X} 2(h_{jk} - \delta_{jk} h_{ii}) * dx_j dvol(X). \end{aligned} \tag{1.8}$$

Here the $*$ operator is the one on the Euclidean factor, the index i, j run over the Euclidean factor while the index a runs over the full index of the manifold.

Then we have

Theorem 1.2 *Assuming that M is spin, one has*

$$E - |P| \geq 0$$

on each end M_l. Moreover, if $E = 0$ for some end M_l, then M has only one end. In this case, when $k = n$, N is flat along M.

In particular, this result includes the original higher dimensional Positive Energy Theorem whose spinor proof is given in [Wi1] and [PT] for dimension 4 and in [Z1] for dimension 5.

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2 The hypersurface Dirac operator

We will adapt Witten's spinor method [Wi1], as given in [PT], to our situation. The crucial ingredient here is the hypersurface Dirac operator on M, acting on the (restriction of the) spinor bundle of N. Let S be the spinor bundle of N and still denote by the same notation its restriction on (or rather, pullback to) M. Denote by ∇ the connection on S induced by the Lorentzian metric on N. The Lorentzian metric on N also induces a Riemannian metric on M, whose Levi-Civita connection gives rise to another connection, $\bar{\nabla}$ on S. The two, of course, differ by a term involving the second fundamental form.

There are two choices of metrics on S, which is another subtlety here. Since part of the treatment in [PT] is special to dimension 4, we will give a somewhat detailed account here.

Let $SO(n,1)$ denote the identity component of the groups of orientation preserving isometries of the Minkowski space $\mathbb{R}^{n,1}$. A choice of a unit timelike covector e^0 gives rise to injective homomorphisms α, $\hat{\alpha}$, and a commutative diagram

$$\begin{array}{cccc} \alpha: & SO(n) & \rightarrow & SO(n,1) \\ & \uparrow & & \uparrow \\ \hat{\alpha}: & Spin(n) & \rightarrow & Spin(n,1). \end{array} \tag{2.9}$$

We now fix a choice of unit timelike normal covector e^0 of M in N. Let $F(N)$ denote the $SO(n,1)$ frame bundle of N and $F(M)$ the $SO(n)$ frame bundle of M. Then $i^*F(N) = F(M) \times_\alpha SO(n,1)$, where $i: M \hookrightarrow N$ is the inclusion. If N is spin, then we have a principal $Spin(n,1)$ bundle $P_{Spin(n,1)}$ on N, whose restriction on M is then $i^*P_{Spin(n,1)} = P_{Spin(n)} \times_{\hat{\alpha}} Spin(n,1)$, where $P_{Spin(n)}$ is the principal $Spin(n)$ bundle of M. Thus, even if N is not spin, $i^*P_{Spin(n,1)}$ is still well-defined as long as M is spin.

Similarly, when N is spin, the spinor bundle S on N is the associated bundle $P_{Spin(n,1)} \times_{\rho_{n,1}} \Delta$, where $\Delta = \mathbb{C}^{2^{[\frac{n+1}{2}]}}$ is the complex vector space of spinors and

$$\rho_{n,1}: \ Spin(n,1) \rightarrow GL(\Delta) \tag{2.10}$$

is the spin representation. Its restriction to M is given by $i^* P_{Spin(n,1)} \times_{\rho_{n,1}} \Delta = P_{Spin(n)} \times_{\rho_n} \Delta$ with

$$\rho_n : \; Spin(n) \overset{\hat{\alpha}}{\hookrightarrow} Spin(n,1) \xrightarrow{\rho_{n,1}} GL(\Delta) \tag{2.11}$$

Again, the restriction is still well defined as long as M is spin.

Let e^0, e^i ($i = 1, \cdots, n$ will be the range for the index i in this section) be an orthonormal basis of the Minkowski space $\mathbb{R}^{n,1}$ of dimension $n+1$ such that $|e^0| = -1$.

Lemma 2.1 *There is a positive definite hermitian inner product* $\langle \; , \; \rangle$ *on* Δ *which is* $Spin(n)$*-invariant. Moreover,* $(s, s') = \langle e^0 \cdot s, s' \rangle$ *defines a hermitian inner product which is also* $Spin(n)$*-invariant but not positive definite. In fact*

$$(v \cdot s, \; s') = (s, \; v \cdot s')$$

for all $v \in \mathbb{R}^{n,1}$.

Proof. Detailed study via Γ matrices [CBDM, p10-11] shows that there is a positive definite hermitian inner product $\langle \; , \; \rangle$ on Δ with respect to which e^i is skew-hermitian while e^0 is hermitian. It follows then that $\langle \; , \; \rangle$ is $Spin(n)$-invariant. We now show that $(s, s') = \langle e^0 \cdot s, s' \rangle$ defines a $Spin(n)$-invariant hermitian inner product. Since e^0 is hermitian with respect to $\langle \; , \; \rangle$, $(\; , \;)$ is clearly hermitian. To show that $(\; , \;)$ is $Spin(n)$-invariant, we take a unit vector v in the Minkowski space: $v = a_0 e^0 + a_i e^i$, $a_0, a_i \in \mathbb{R}$ and $-a_0^2 + \sum_{i=1}^n a_i^2 = 1$. Then

$$\begin{aligned}
(vs, vs') &= \langle e^0 vs, vs' \rangle \\
&= a_0^2 \langle e^0 e^0 s, e^0 s' \rangle + a_i a_0 \langle e^0 e^i s, e^0 s' \rangle + a_0 a_i \langle e^0 e^0 s, e^i s' \rangle + a_i a_j \langle e^0 e^i s, e^j s' \rangle \\
&= a_0^2 \langle s, e^0 s' \rangle - a_i a_j \langle e^j e^0 e^i s, s' \rangle \\
&= a_0^2 \langle e^0 s, s' \rangle + a_i a_j \langle e^0 e^j e^i s, s' \rangle \\
&= a_0^2 \langle e^0 s, s' \rangle - a_i^2 \langle e^0 s, s' \rangle \\
&= -(s, s')
\end{aligned}$$

Consequently, $(\; , \;)$ is $Spin(n)$-invariant. The above computation also implies that $v\cdot$ acts as hermitian operator on Δ with respect to $(\; , \;)$. ■

Thus the spinor bundle S restricted to M inherits an hermitian metric $(\; , \;)$ and a positive definite metric $\langle \; , \; \rangle$. They are related by the equation

$$(s, s') = \langle e^0 \cdot s, s' \rangle. \tag{2.12}$$

Now the hypersurface Dirac operator is defined by the composition

$$\mathcal{D} : \; \Gamma(M, S) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes S) \xrightarrow{c} \Gamma(M, S), \tag{2.13}$$

where c denotes the Clifford multiplication. In terms of a local orthonormal basis $e_1, e_2, \cdots, e_n$ of TM,

$$\mathcal{D}\psi = e^i \cdot \nabla_{e_i} \psi,$$

where e^i denotes the dual basis.

The two most important properties of hypersurface Dirac operator are the self-adjointness with respect to the metric $\langle \; , \; \rangle$ and the Bochner-Lichnerowicz-Weitzenbock formula [Wi1], [PT].

Lemma 2.2 *Define a $n-1$ form on M by $\omega = \langle \phi, e^i \cdot \psi \rangle \mathrm{int}(e_i)\, dvol$, where $dvol$ is the volume form of the Riemannian metric g. We have*

$$[\langle \phi, \mathcal{D}\psi \rangle - \langle \mathcal{D}\phi, \psi \rangle] dvol = d\omega.$$

Thus $\mathcal{D}$ is formally self adjoint with respect to the L^2 metric defined by $\langle\ ,\ \rangle$ (and $dvol$).

Proof. Since ω is independent of the choice of the orthonormal basis, we do our computation locally using a preferred basis. For any given point $p \in M$, choose a local orthonormal frame e_i of TM near p such that $\bar{\nabla} e_i = 0$ at p. Extend e_0, e_i to a neighborhood of p in N by parallel translating along e_0 direction. Then, at p, $\nabla_{e_i} e^j = -h_{ij} e^0$ and $\nabla_{e_i} e^0 = -h_{ij} e^j$. Therefore (again at p),

$$\begin{aligned}
d\omega &= \nabla_{e_i} \langle \phi, e^i \cdot \psi \rangle \, dvol \\
&= [((\nabla_{e_i} e^0) \cdot \phi, e^i \cdot \psi) + (e^0 \cdot \nabla_{e_i} \phi, e^i \cdot \psi) + (e^0 \cdot \phi, (\nabla_{e_i} e^i) \cdot \psi) + (e^0 \cdot \phi, e^i \cdot \nabla_{e_i} \psi)] dvol \\
&= [-h_{ij}(e^j \cdot \phi, e^i \cdot \psi) + (e^i \cdot e^0 \cdot \nabla_{e_i} \phi, \psi) - h_{ii}(e^0 \cdot \phi, e^0 \cdot \psi) + \langle \phi, \mathcal{D}\psi \rangle] dvol \\
&= [-h_{ij}(e^i \cdot e^j \cdot \phi, \psi) - \langle \mathcal{D}\phi, \psi \rangle - h_{ii}(e^0 \cdot \phi, e^0 \cdot \psi) + \langle \phi, \mathcal{D}\psi \rangle] dvol \\
&= [-\langle \mathcal{D}\phi, \psi \rangle + \langle \phi, \mathcal{D}\psi \rangle] dvol
\end{aligned}$$

■

Now the Bochner-Lichnerowicz-Weitzenbock formula.

Lemma 2.3 *One has*

$$\begin{aligned}
\mathcal{D}^2 &= \nabla^* \nabla + \mathcal{R}, \\
\mathcal{R} &= \frac{1}{4}(R + 2R_{00} + 2R_{0i} e^0 \cdot e^i \cdot) \in End(S).
\end{aligned} \tag{2.14}$$

Here the adjoint ∇^* is with respect to the metric $\langle\ ,\ \rangle$.

Proof. We again do the computation in the frame as in the proof of Lemma 2.2. Then

$$\begin{aligned}
\mathcal{D}^2 &= e^i \cdot e^j \cdot \nabla_{e_i} \nabla_{e_j} + e^i \cdot \nabla_{e_i} e^j \cdot \nabla_{e_j} \\
&= -\nabla_{e_i} \nabla_{e_i} + \frac{1}{4}(R + 2R_{00} + 2R_{0i} e^0 \cdot e^i \cdot) - h_{ij} e^i \cdot e^0 \cdot \nabla_{e_j}.
\end{aligned}$$

Now

$$\begin{aligned}
d[\langle \phi, \psi \rangle \mathrm{int}(e_i)\, dvol] &= e_i \langle \phi, \psi \rangle \, dvol \\
&= (\nabla_{e_i} e^0 \cdot \phi, \psi) + \langle \nabla_{e_i} \phi, \psi \rangle + \langle \phi, \nabla_{e_i} \psi \rangle \\
&= -h_{ij}(e^j \cdot \phi, \psi) + \langle \nabla_{e_i} \phi, \psi \rangle + \langle \phi, \nabla_{e_i} \psi \rangle \\
&= -h_{ij} \langle e^0 \cdot e^j \cdot \phi, \psi \rangle + \langle \nabla_{e_i} \phi, \psi \rangle + \langle \phi, \nabla_{e_i} \psi \rangle
\end{aligned}$$

This shows that $\nabla^*_{e_i} = -\nabla_{e_i} - h_{ij} e^j \cdot e^0 \cdot$. The desired formula follows. ■

3 Proof of the Theorem

By the Einstein equation,

$$\mathcal{R} = 4\pi(T_{00} + T_{0i}e^0 \cdot e^i \cdot).$$

It follows then from the dominant energy condition (1.4) that

$$\mathcal{R} \geq 0. \tag{3.15}$$

Now, for $\phi \in \Gamma(M, S)$ and a compact domain $\Omega \subset M$ with smooth boundary, the Bochner-Lichnerowicz-Weitzenbock formula yields

$$\begin{aligned} \int_\Omega [|\nabla\phi|^2 + \langle\phi, \mathcal{R}\phi\rangle - |\mathcal{D}\phi|^2]\, dvol(g) &= \int_{\partial\Omega} \sum \langle(\nabla_{e_a} + e_a \cdot \mathcal{D})\phi,\ \phi\rangle \,\mathrm{int}(e_a)\, dvol(g) \quad (3.16) \\ &= \int_{\partial\Omega} \sum \langle(\nabla_\nu + \nu \cdot \mathcal{D})\phi,\ \phi\rangle\, dvol(g|_{\partial\Omega}), \quad (3.17) \end{aligned}$$

where e_a is an orthonormal basis of g and ν is the unit outer normal of $\partial\Omega$. Also, here $\mathrm{int}(e_a)$ is the interior multiplication by e_a.

Now let the manifold $M = M_0 \cup M_\infty$ with M_0 compact and $M_\infty \simeq (\mathbb{R}^k - B_R(0)) \times X$, and (X, g_X) a compact Riemannian manifold with nonzero parallel spinors. Moreover, the metric g on M satisfies (1.6). Let e_a^0 be the orthonormal basis of $\mathring{g}$ which consists of $\frac{\partial}{\partial x_i}$ followed by an orthonormal basis f_α of g_X. Orthonormalizing e_a^0 with respect to g gives rise an orthonormal basis e_a of g. Moreover,

$$e_a = e_a^0 - \frac{1}{2}u_{ab}e_b^0 + O(r^{-2\tau}). \tag{3.18}$$

This gives rise to a gauge transformation

$$A:\ SO(\mathring{g}) \ni e_a^0 \to e_a \in SO(g)$$

which identifies the corresponding spin groups and spinor bundles.

We now pick a unit norm parallel spinor ψ_0 of $(\mathbb{R}^k, g_{\mathbb{R}^k})$ and a unit norm parallel spinor ψ_1 of (X, g_X). Then $\phi_0 = A(\psi_0 \otimes \psi_1)$ defines a spinor of M_∞. We extend ϕ_0 smoothly inside. Then $\nabla^0\phi_0 = 0$ outside the compact set.

Lemma 3.1 *If a spinor ϕ is asymptotic to ϕ_0: $\phi = \phi_0 + O(r^{-\tau})$, then we have*

$$\lim_{R\to\infty} \Re \int_{S_R\times X} \sum\langle(\nabla_{e_a} + e_a \cdot D)\phi,\ \phi\rangle\, \mathrm{int}(e_a)\, dvol(g) = \omega_k vol(X)\langle\phi_0,\ E\phi_0 + P_k dx^0 \cdot dx^k \cdot \phi_0\rangle,$$

where $\Re$ means taking the real part.

Proof. Recall that $\bar\nabla$ denote the connection on S induced from the Levi-Civita connection on M. We have

$$\nabla_{e_a}\psi = \bar\nabla_{e_a}\psi - \frac{1}{2}h_{ab}e^0 \cdot e^b \cdot \psi. \tag{3.19}$$

By the Clifford relation,

$$\langle (\nabla_{e_a} + e_a \cdot D)\phi,\ \phi\rangle = -\frac{1}{2}\langle [e^a\cdot, e^b\cdot]\nabla_{e_b}\phi,\ \phi\rangle.$$

Hence

$$\int_{S_R\times X} \sum \langle (\nabla_{e_a} + e_a \cdot D)\phi,\ \phi\rangle \operatorname{int}(e_a)\, dvol(g) =$$
$$-\frac{1}{2}\int_{S_R\times X} \langle [e^a\cdot, e^b\cdot]\bar{\nabla}_{e_b}\phi,\ \phi\rangle \operatorname{int}(e_a)\, dvol(g) + \frac{1}{4}\int_{S_R\times X} \langle [e^a\cdot, e^b\cdot] h_{bc} e^0 \cdot e^c \cdot \phi,\ \phi\rangle \operatorname{int}(e_a)\, dvol(g).$$

Using (3.18) and the asymptotic conditions (1.7), the second term in the right hand side can be easily seen to give us

$$\lim_{R\to\infty} \frac{1}{4}\int_{S_R\times X} \langle 2(h_{ac} - \delta_{ac}h_{bb})e^0 \cdot e^c \cdot \phi,\ \phi\rangle \operatorname{int}(e_a)\, dvol(g) = \omega_k vol(X)\langle \phi_0,\ P_k dx^0 \cdot dx^k \cdot \phi_0\rangle.$$

The first term is computed in [D] to limit to

$$\omega_k vol(X)\langle \phi_0,\ E\phi_0\rangle.$$

■

The following lemma is standard [PT], [Wi1].

Lemma 3.2 *If*

$$\langle \phi_0,\ E\phi_0 + P_k dx^0 \cdot dx^k \cdot \phi_0\rangle \geq 0$$

for all constant spinors ϕ_0, then

$$E - |P| \geq 0.$$

As usual, the trick to get the positivity now is to find a harmonic spinor ϕ asymptotic to ϕ_0. Then the left hand side of (3.16) will be nonnegative since $\mathcal{R} \geq 0$. Passing to the right hand side will give us the desired result.

Lemma 3.3 *There exists a harmonic spinor ϕ on $(M,\ g)$ which is asmptotic to the parallel spinor ϕ_0 at infinity:*

$$\mathcal{D}\phi = 0, \quad \phi = \phi_0 + O(r^{-\tau}).$$

Proof. The proof is essentially the same as in [D]. We use the Fredholm property of $\mathcal{D}$ on a weighted Sobolev space and $\mathcal{R} \geq 0$ to show that it is an isomorphism. The harmonic spinor ϕ can then be obtained by setting $\phi = \phi_0 + \xi$ and solving $\xi \in O(r^{-\tau})$ from the equation $\mathcal{D}\xi = -\mathcal{D}\phi_0$. ■

The rest of the Theorem follows as in [PT].

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